PRIIMEK	IME	VPISNA ŠTEVILKA	SMER

NALOGA	TOČKE
1.	
2.	
3.	
4.	
SKUPAJ	

RAČUNSKI DEL IZPITA IZ MATEMATIČNE ANALIZE II (11.9.2006)

Navodilo: vsako nalogo rešuj na strani, kjer je napisana. Če bo naloga reševana kje drugje, mora biti to posebej označeno. Veliko sreče pri reševanju!

1. naloga: Izračunaj vrednost $\int_0^\infty x^{2m}e^{-x^2}\,dx$. Kolikšna je vrednost za $m=\frac{5}{2}$?

Nasvet: Uporabi funkcijo gama: $\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}\,dx$, $\Gamma(s+1)=s\Gamma(s)$; $s\in\mathbb{R}$, $\Gamma(1)=1$, $\Gamma(n+1)=n!$; $n\in\mathbb{N}$, $\Gamma(\frac{1}{2})=\sqrt{\pi}$, $\Gamma(p)\Gamma(1-p)=\frac{\pi}{\sin p\pi}$. (30 točk)

25

$$\int_0^\infty x^{2m}e^{-x^2}\,dx=\int_0^\infty t^{\frac{2m}{2}}\cdot e^{-t}\,\frac{t^{-\frac{1}{2}}dt}{2}=$$

$x^2=t$ $\quad\alpha=0$

$2x\,dx=dt$ $\quad\beta=\infty$

$dx=\frac{dt}{2x}=\frac{dt}{2\cdot\sqrt{t}}=\frac{t^{-\frac{1}{2}}dt}{2}$

$$=\frac{1}{2}\cdot\int_0^\infty t^{m-\frac{1}{2}}e^{-t}\,dt=*$$

$s-1=m-\frac{1}{2}$

$s=m+\frac{1}{2}$

$$*=\boxed{\Gamma\left(m+\frac{1}{2}\right)}$$

$m=\frac{5}{2}$

$$\Rightarrow\int_0^\infty x^5e^{-x^2}\,dx=\Gamma\left(\frac{5}{2}+\frac{1}{2}\right)=\Gamma(3)=2\cdot 1=\boxed{2}$$

2. naloga: Za funkcijo $f(x) = e^x$ zapiši Taylorjevo formulo reda 4. Z njeno pomočjo izračunaj približno vrednost integrala $\int_0^1 e^{-x^2}dx$ in oceni napako, ki jo pri tem narediš. (25 točk)

$f(x) = e^x$

$f'(x) = e^x = \dots f^{(IV)}(x)$

$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + r_4$

$r_4 = \frac{e^{\Theta x}}{5!} \cdot x^5, \Theta \in [0,1]$

$e^{-x^2} = f(-x^2) = 1 - \frac{x^2}{1} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + r_4^*$

$r_4^* = -\frac{e^{-\Theta x^2}}{5!} \cdot x^{10}, \Theta \in [0,1]$

$\int_0^1 e^{-x^2}dx = \int_0^1 (1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} + r_4^*)\,dx \doteq$

$\doteq \left(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!}\right)\Big|_0^1 =$

$= \boxed{1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!}}$

Ocena napake:

$\left|\int_0^1 r_4^* dx\right| = \left|\int_0^1 \left(-\frac{e^{-\Theta x^2}}{5!} \cdot x^{10}\right) dx\right| \leq$

$\leq \int_0^1 \frac{x^{10}}{5!}\,dx = \left(\frac{x^{11}}{11 \cdot 5!}\right)\Big|_0^1 = \boxed{\frac{1}{11 \cdot 5!}}$

3. naloga: Izračunaj konvergenčni polmer vrste in jo seštej: $f(z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2n+1}}{2n-1}$. (25 točk)

• kvocientni kriterij

$$q = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{z^{2n+3}}{2n+1}}{\frac{z^{2n+1}}{2n-1}}\right| = \frac{2n-1}{2n+1}\cdot z^2$$

$$\lim_{n\to\infty} q = z^2 \lim_{n\to\infty}\frac{2n-1}{2n+1} = z^2, \quad z^2 < 1 \Leftrightarrow |z| < 1$$

Konvergenčni polmer je 1.

• $S(z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2n+1}}{2n-1} =$

$$= \frac{z^3}{1} - \frac{z^5}{3} + \frac{z^7}{5} - \dots =$$

$$= z^2\cdot\underbrace{\left(z - \frac{z^3}{3} + \frac{z^5}{5} - \dots\right)}_{v(z)}$$

$$v'(z) = \underbrace{1 - z^2 + z^4 - \dots}_{\text{geo. vrsta}, \ |q| = |-z^2| < 1} = \frac{1}{1+z^2}$$

$$v(z) = \int_0^z \frac{1}{1+t^2}\,dt = \operatorname{arctg} t \Big|_0^z = \operatorname{arctg} z$$

$$S(z) = z^2\cdot v(z) = \boxed{z^2\cdot\operatorname{arctg} z}$$

4. naloga: Za pozitivna števila x, y in z velja $x + y + z = 1$. Določi njihovo vrednost tako, da bo njihov produkt maksimalen. (~~20~~ 25 točk)

$x, y, z > 0$

$x+y+z=1 \Leftrightarrow x+y+z-1=0$

$g(x,y,z) = x+y+z-1$

Metoda vezanih ekstremov:

$P(x,y,z) = x \cdot y \cdot z \leftarrow$ iščemo max., na $g(x,y,z)=0$

$F(x,y,z,\lambda) = xyz + \lambda(x+y+z-1)$

$\frac{\partial F}{\partial x} = yz + \lambda = 0 \qquad y = -\frac{\lambda}{z} \quad (z \neq 0)$

$\frac{\partial F}{\partial y} = xz + \lambda = 0 \qquad x = -\frac{\lambda}{z} \qquad \boxed{x=y}$

$\frac{\partial F}{\partial z} = xy + \lambda = 0$

$\frac{\partial F}{\partial \lambda} = x+y+z-1 = 0 \qquad y = -\frac{\lambda}{x} \qquad z = -\frac{\lambda}{x} \quad (x \neq 0)$

$\boxed{y = z}$

$x + x + x = 1$

$\boxed{x = \frac{1}{3} = y = z}$